Lecture 15

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Welcome to the 305

Goal Programming

- All prior linear programming problems have had a single objective function
- Companies may have multiple criteria in consideration for a decision
- Sometimes the multiple objectives conflict
- Company may want to maximize profit and minimize pollution
- Goal programming is linear programming for multiple objectives or criteria

Ex: Beaver Creek Pottery

- Trying to choose $x = number\ of\ bowls$ and y $= number\ of\ mugs$ to maximize the profit function

- Recall the original linear program

Maximize $\quad 40x + 50y$

Subject to: $\quad x + 2y \leq 40 \quad$ (Labor)

$4x + 3y \leq 120 \quad$ (Clay)

$x, y \geq 0$

- Objective function reflects a single goal

Ex: Beaver Creek Pottery

- Suppose Beaver Creek wanted to achieve other goals while maximizing profit

- In order of importance:
  - To avoid layoffs, they want to use at least 40 hours of labor per day
  - They want to achieve a satisfactory profit level of $1,600 per day
  - To avoid having clay dry out, they prefer to keep no more than 120 lb of clay on hand each day
  - To avoid overhead costs due to keeping the factory open past normal hours, they want to minimize the amount of overtime

- We reformulate our linear programming model using goal programming

- Transform linear programming model constraints into goals

Ex: Beaver Creek Pottery

- Goal 1: Avoid underutilization of labor
  - Original constraint $x + 2y \leq 40$
  - Reformulation to a goal constraint

$$x + 2y + d_1^- - d_1^+ = 40 \qquad \text{(Labor)}$$

  - Two new variables d_1^- and d_1^+ are nonnegative and represent the underutilized time and overtime, respectively
  - Q: What if the optimal solution had $d_1^- > 0$?
  - Q: What if the optimal solution had $d_1^+ > 0$?
  - The top priority corresponding to minimization of d_1^-

Minimize $\quad P_1 d_1^-$

  - The P_1 indicates the priority of this goal (not a coefficient)

Ex: Beaver Creek Pottery

- Goal 2: Achieve daily profit of $1,600
  - Original objective function $Z = 40x + 50y$
  - Reformulation to a goal constraint

$$40x + 50y + d_2^- - d_2^+ = 1600 \qquad \text{(Profit)}$$

  - Two new variables d_2^- and d_2^+ are nonnegative and represent the amount of profit less than $1,600 and more than $1,600
  - The second priority corresponding to minimization of d_2^- is added

Minimize $\quad P_1 d_1^-, P_2 d_2^-$

  - The comma between the terms indicates that we are minimizing them sequentially, not simultaneously
  - Q: Why are we not minimizing d_2^+?

Ex: Beaver Creek Pottery

- Goal 3: Avoid waste of material
  - Original constraint $4x + 3y \leq 120$
  - Reformulation to a goal constraint

$$4x + 3y + d_3^- - d_3^+ = 120 \qquad \text{(Clay)}$$

  - Two new variables d_3^- and d_3^+ are nonnegative and represent the amount of clay less than 120 lbs and more than 120 lbs
  - The company cannot keep more than 120 lbs in storage
  - The third priority corresponds to minimization of d_3^+ is added

Minimize $\quad P_1 d_1^-, P_2 d_2^-, P_3 d_3^+$

Ex: Beaver Creek Pottery

- Goal 4: Avoid overtime costs
  - Recall the modified goal constraint for labor

$$x + 2y + d_1^- - d_1^+ = 40 \qquad \text{(Labor)}$$

  - Already attempting to minimize d_1^-
  - To ensure we don't exceed the maximum labor, we involve d_1^+
  - Finalization of objective function

Minimize $\quad P_1 d_1^-, P_2 d_2^-, P_3 d_3^+, P_4 d_1^+$

Ex: Beaver Creek Pottery

- Full goal programming model

Minimize $P_1 d_1^-, P_2 d_2^-, P_3 d_3^+, P_4 d_1^+$

Subject to

$$
\begin{aligned}
x + 2y + d_1^- - d_1^+ &= 40 && \text{(Labor)} \\
40x + 50y + d_2^- - d_2^+ &= 1600 && \text{(Profit)} \\
4x + 3y + d_3^- - d_3^+ &= 120 && \text{(Clay)} \\
x, y, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+ &\geq 0
\end{aligned}
$$

- The variables $\{d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+\}$ are called deviational variables
- We minimize the four different objective functions individually by priority

Ex: Beaver Creek Pottery

- Modification 1: Prefer no more than 10 hours of overtime
  - Recall the goal constraint for labor

$$x + 2y + d_1^- - d_1^+ = 40 \qquad \text{(Under hours)}$$

  - Remember that d_1^+ represents overtime
  - We want $0 \leq d_1^+ \leq 10$
  - Use same strategy as before by adding a goal constraint

$$d_1^+ + d_4^- - d_4^+ = 10 \qquad \text{(Over hours)}$$

  - Possible goal constraint of all deviational variables
  - Two new variables d_4^- and d_4^+ are nonnegative and represent the amount of overtime hours less than 10 hours and more than 10 hours
  - New objective function

Minimize $\quad P_1 d_1^-, P_2 d_2^-, P_3 d_3^+, P_4 d_4^+$

Ex: Beaver Creek Pottery

- Modification 2: Maximum number of bowls and mugs made daily
  - Pottery company has limited warehouse space
  - They can only store at most 30 bowls and 20 mugs each day
  - Profit for bowls (\$40) less than profit for mugs (\$50)
  - Consider the new constraints

$$x + d_5^- = 30 \qquad \text{(Bowls)}$$
$$y + d_6^- = 20 \qquad \text{(Mugs)}$$

  - We want to minimize d_5^- and d_6^-
  - Q: Why not include positive deviational variables d_5^+ and d_6^+ ?
  - Q: For which item is it more important to achieve this goal?

Ex: Beaver Creek Pottery

- Modification 2: Maximum number of bowls and mugs made daily
  - Positive deviational variables are unnecessary since it is imperative to not exceed the warehouse space
  - We need to achieve the goal for mugs more than the goal for bowls because the profit is higher for mugs
  - If goals were of equal importance, we would minimize

Minimize $$P_1 d_1^-, P_2 d_2^-, P_3 d_3^+, P_4 d_4^+, P_5 (d_5^- + d_6^-)$$

  - We can make the degree of importance in proportion to the profit
  - The goal for mugs is more important than the goal for bowls by a ratio of 5 to 4

Minimize $$P_1 d_1^-, P_2 d_2^-, P_3 d_3^+, P_4 d_4^+, P_5 (4d_5^- + 5d_6^-)$$

  - The coefficients 4 and 5 are referred to as weights

Ex: Beaver Creek Pottery

- Full modified goal programming model

Minimize $P_1 d_1^-, P_2 d_2^-, P_3 d_3^+, P_4 d_4^+, P_5(4d_5^- + 5d_6^-)$

Subject to

$$
\begin{array}{ll}
x + 2y + d_1^- - d_1^+ = 40 & \text{(Labor)} \\
40x + 50y + d_2^- - d_2^+ = 1600 & \text{(Profit)} \\
4x + 3y + d_3^- - d_3^+ = 120 & \text{(Clay)} \\
d_1^+ + d_4^- - d_4^+ = 10 & \text{(Overtime)} \\
x + d_5^- = 30 & \text{(Bowls)} \\
y + d_6^- = 20 & \text{(Mugs)} \\
x, y, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+, d_4^-, d_4^+, d_5^-, d_6^- \geq 0 &
\end{array}
$$

The End

Dale